Study of a function

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1 Algorithm

In this brief exposition the aim is to set a complete algorithm to study a one dimensional real valued function

$$f : D_f \subseteq \mathbb{R} \to \mathbb{R} \qquad x \mapsto f(x) \tag{1}$$

Note that in expression (1) D_f denotes the *domain* of the function f, that can be either a subset (or the union of subsets) of $\mathbb{R}$ or the whole real line. In order to solve completely the study of a function and draw its graph the following basics concepts and techniques are fundamental (and it is strongly suggested to review the main definitions and theorems before proceeding):

- Solving inequalities (fractional, logarithmic and with radicals in particular).
- Definitions of monotonicity and convexity of a function. What are the tools used to study monotonicity and convexity?
- Notions of limit and asymptote (vertical, horizontal, oblique). Where is it useful to calculate limits of a function?
- Concepts of continuity and differentiability of a function. What are the three possibile discontinuity points? What are the possible points of non-differentiability?

The following list enumerates the steps of the study of a function.

1. **Domain.**
   To find the domain D_f it's fundamental remember how to solve inequalities. Moreover, one has to recall that basically only three conditions must be checked and put together in a system (and remember that solving a system means to find the common solutions of the equalities or inequalities that constitute the system itself).
   - *Denominators:* they must be **non-zero**.
   - *Radicals (with even index)*: the argument (that is the expression under the sign of root) must be **greater or equal to zero.**
   - *Logarithms:* the argument (that is the expression inside the sign of logarithm) must be **strictly greater than zero.**
2. **Simmetries.**
   To find simmetries in the function is useful to draw the graph in a easiest way. Basically there are two kinds of simmetries:

- *Simmetry with respect the y-axis:* if $f(-x) = f(x)$ the function is said to be **even** and it is sufficient to draw the graph considering only the half plane $x > 0$: the other half will be the reflection of this graph with respect to the vertical axis.
- *Simmetry with respect the origin:* if $f(-x) = -f(x)$ the function is said to be **odd** and it is sufficient to draw the graph considering only the half plane $x > 0$: the other half will be the reflection of this graph with respect to the origin.

3. **Sign of the function.**
Once that domain and simmetries have been found one can begin to sketch the graph of the function by studying its sign. Basically the inequality

$$f(x) \geq 0 \tag{2}$$

must be solved. Remember that inequality (2) *must be always solved* ***inside*** *the domain* D_f so actually to study the sign the correct system to solve is the following:

$$\begin{cases} f(x) \geq 0 \\ x \in D_f \end{cases}$$

In order to determine the sign it's useful to remember the following fact: *in the domain* D_f

- radicals with even indexes are always positive (and they are zero if the argument is zero);
- exponverticalential are always positive.

Recall that the expression *in the domain* D_f means that if a function is for instance

$$f(x) = \sqrt{x-1}$$

it is not true that f is positive always (namely for every value of x): the right thing is that f is always positive in the domain, that is $D_f = [1, +\infty)$ (so actually f is not positive for each x but only for the ones contained in D_f!).

4. **Intersection with the axis.**
Observe that by studying the sign of a function one finds also the intersection with the $x-$axis (that is where $f(x) = 0$). It's useful to determine the intersection with the $y-$axis too, by substituting $x = 0$ in the expression of f (of course **only** if $x = 0$ belongs to the domain, otherwise it's just wasting time!).

5. **Limits and asymptotes.**
Limits have to be calculated on the *critical points* of the domains (that is, the *end points* of the intervals that make up the domain). For instance, if the domain is of the form

$$D_f = (-\infty; 2] \cup (1, +\infty)$$

limits must be calculated for $x \to -\infty$, $x \to +\infty$ (because they are in some ways included in D_f!) and for $x \to 1^+$. Note that around 1 we only need the right limit since x is not allowed to take values lesser than 1 and so the limit coming from the left does not make any sense). Moreover, we *do not need* the limit at $x = 2$ since this value is *included* in D_f and thus here the function is continuous and the limit is equal to $f(2)$. Recall the following facts:

- $\lim_{x\to x_0} f(x) = \pm\infty$ means that at $x = x_0$ (where x_0 is a *finite* number) the function has a **vertical asymptotes**. Actually one has to calculate both the left and the right limit since the may be different.
- $\lim_{x\to\pm\infty} f(x) = L < \infty$ means that for $x \to \pm\infty$ the function has a **horizontal asymptote** whose equation is $y = L$.
- $\lim_{x\to\pm\infty} f(x) = \pm\infty$ means that the function is unbounded and grows to infinity. In this case there are three possibile ways of growing:
  - *Superlinear:* the function goes to infinity faster than any straight line (e.g. the exponential $y = e^x$).
  - *Sublinear:* the function goes to infinity slower than any stright line (e.g. the logarithm $y = \ln x$).
  - *Linear:* in this case the function has an **oblique asymptote** of equation $y = mx + q$ where

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad q = \lim_{x\to\pm\infty} [f(x) - mx]$$

6. **Monotonicity.**

Differential calculus comes now into play. To study weather the function *increases* or *decreases* we need to compute the **first derivative** $f'(x)$ and determine its sign. The following **monotonicity test** holds:

- $f'(x) > 0 \quad \Rightarrow \quad f$ increases.
- $f'(x) < 0 \quad \Rightarrow \quad f$ decreases.

Where $f'(x) = 0$ the question is more delicate. Recall that if $x_0 \in D_f$ is such that $f'(x_0) = 0$ then x_0 is called **stationary point** for f, meaning that the tangent line to the graph of f *at the point* x_0 is horizontal. But in this case x_0 can have different natures.

- If f increases before x_0 and decreases after x_0, that is

$$\begin{cases} f'(x) > 0 & x < x_0 \\ f'(x) < 0 & x > x_0 \end{cases}$$

  then x_0 is a **maximum point**. Pay attention to the fact that if you are asked to find the *maximum* of the function f, this is $f(x_0)$ and *not* x_0 (which is actually the point of the domain that let the maximum of f to be achieved).

- In the same way we can define the minimum point. If f decreases before x_0 and increases after x_0, that is

$$\begin{cases} f'(x) < 0 & x < x_0 \\ f'(x) > 0 & x > x_0 \end{cases}$$

  then x_0 is a **minimum point**. As before, the *minimum* of the function must be calculated and it is $f(x_0)$.

- If the first derivative does not change sign passing through x_0, that is f increases (or decreases) both before and after x_0, then x_0 is called **inflection point with horizontal tangent**. A relevant function exhibiting such a charachteristic is the cubic line $f(x) = x^3$: since $f'(x) = 3x^2$ is always positive and it is zero only for $x_0 = 0$, the cubic has an inflection point at the origin.

Summing up: at this step just compute the first derivative of f and determine its sign: in one shot, writing a suitable table, you'll be able to find the stationary points and the monotonicity of the function.

7. **Convexity.**
Last step before drawing the graph: compute the **second derivative** $f''(x)$ and determine its sign. In this case the test based on the sign of the second derivative is usueful to define convexity of the functiom. Briefly:

- $f''(x) > 0 \quad \Rightarrow \quad f$ is convex.
- $f''(x) < 0 \quad \Rightarrow \quad f$ is concave.

If at a point $x_0 \in D_f$ the second derivative is zero, $f''(x_0) = 0$, then the function exhibit a change in convexity at x_0 and the point is called **inflection point** (with **oblique tangent**).

2 Examples

To implement the machinery we have just built up we show two examples.

2.1 Example 1

Consider the function

$$f(x) = \frac{x-1}{x^2-x-6}$$

1. **Domain.**
The only condition we have to impose is that the denominator must be non-zero. Therefore solving

$$x^2 - x - 6 \neq 0 \quad \Rightarrow \quad x \neq \frac{1 \pm \sqrt{1+24}}{2} \quad \begin{cases} x \neq 3 \\ x \neq -2 \end{cases}$$

we end up with the following expression for the domain:

$$D_f = \mathbb{R} \backslash \{-2, 3\} \qquad \text{or} \qquad D_f = (-\infty, -2) \cup (-2, 3) \cup (3, +\infty)$$

(but note how the first notation is way more compact and intuitive).

2. **Simmetries.**
Calculating $f(-x)$ (that is replacing all the x's in the expression of f with $-x$) we have

$$f(-x) = \frac{-x-1}{(-x)^2 - (-x) - 6} = \frac{-x-1}{x^2+x-6}$$

that is obviously different from $f(x)$ and $-f(x)$ too. So f is neither even nor odd and does not exhibit simmetries.

3. **Sign of the function.**
Basically we need to solve the inequality

$$f(x) \geq 0 \qquad \frac{x-1}{x^2-x-6} \geq 0$$

inside the domain D_f (note that the condition to remain inside the domain is in this case automatically satisfied because while solving the inequality we'll again impose the denominator to be non-zero). Study separately the sign of the numerator and the denominator (observe that we let N to be zero but *not* D!):

$$N \geq 0 \qquad x-1 \geq 0 \quad \Rightarrow \quad x \geq 1$$
$$D > 0 \qquad x^2-x-6 > 0 \quad \Rightarrow \quad x < -2 \vee x > 3$$

and summarize these information in a table in order to determine the sign of the *whole* fraction.

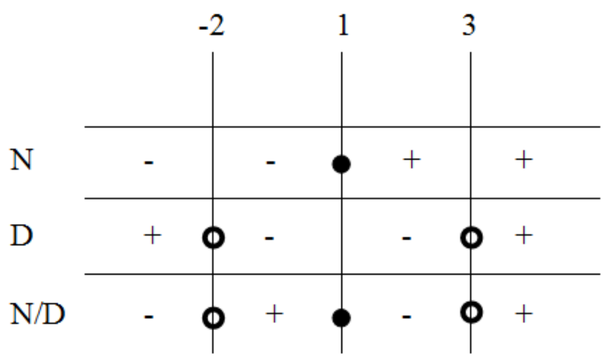

Figure 1:

Therefore, the function f is

- positive in $(-2, 1] \cup (3, +\infty)$
- negative in $(-\infty, -2) \cup [1, 3)$

Note that in the above figure the full circle means that the value $x = 1$ is included in the domain (as expected; moreover, it is exactly the value such that f is zero), while the empty circle indicates values excluded from the domain.

4. **Intersection with the axis.**
Thanks to the previous calculation we have already found the (only) intersection with the $x-$axis: the point $(1, 0)$. We can also find the intersection with the $y-$axis by substituting $x = 0$ (that is included in the domain) in the expression of f and obtaining the point $\left(0, \frac{1}{6}\right)$.

5. **Limits and asymptotes.**
We have six limits to calculate. Note first that since we have excluded two points from the domain because of the denominator, it's likely to expect that the function exhibit vertical asymptotes at these

points. Actually we have:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{x-1}{x^2-x-2} = \left[\frac{-3}{-5\cdot 0^-}\right] = -\infty$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x-1}{x^2-x-2} = \left[\frac{-3}{-5\cdot 0^+}\right] = +\infty$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to -3} \frac{x-1}{x^2-x-2} = \left[\frac{2}{5\cdot 0^-}\right] = -\infty$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x-1}{x^2-x-2} = \left[\frac{-3}{-5\cdot 0^+}\right] = +\infty$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{x-1}{x^2-x-2} = 0$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x-1}{x^2-x-2} = 0$$

Thus we can conclude that the function has two **vertical asymptotes** at $x = -2$ and $x = 3$ and a **horizontal asymptote** of equation $y = 0$ as $x \to \pm\infty$.

6. **Monotonicity (first derivative).**
We have

$$f'(x) = \frac{1\cdot(x^2-x-6)-(x-1)(2x-1)}{(x^2-x-6)^2} = \frac{x^2-x-6-2x^2+x+2x-1}{(x^2-x-6)^2} = \frac{-x^2+2x-7}{(x^2-x-6)^2}$$

and since $-x^2+2x-7$ has $\Delta = 4-28 < 0$ the numerator is always negative (because the coefficient of the second order term is negative!). Thus overall the first derivative is always negative and the function is always decreasing in its domain. The function does not exhibit maxima or minima since f' is never zero.

7. **Convexity (second derivative).**
We have

$$\begin{aligned} f''(x) &= \frac{(-2x+2)(x^2-x-6)^2-2(-x^2+2x-7)(x^2-x-6)(2x-1)}{(x^2-x-6)^4} \\ &= \frac{(x^2-x-6)(-2x^3+2x^2+12x+2x^2-2x-12+4x^3-2x^2-8x^2+4x+28x-14)}{(x^2-x-6)^4} \\ &= \frac{2x^3-6x^2+42x-26}{(x^2-x-6)^3} \end{aligned}$$

Compute the sign of this expression is rather difficult but with some calculation we can find that the only root of the numerator is

$$x_0 = 1+\sqrt[3]{12}-\sqrt[3]{18} \approx 0,67$$

So the numerator is positive if $x > x_0$. Moreover, since the denominator is positive if $x < -2$ or $x > 3$ (same calculus of point (3)) we end up with the following table of the sign of the second derivative:

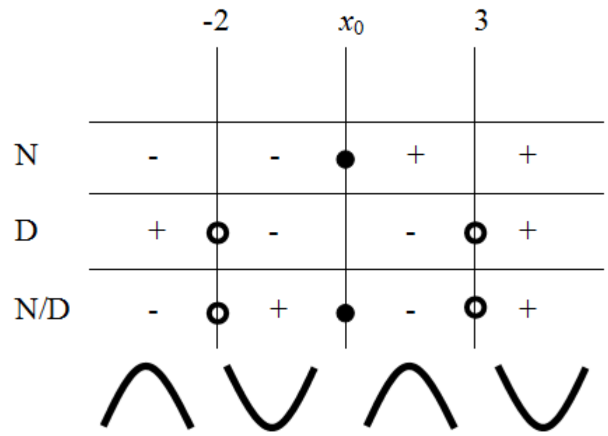

Figure 2:

Therefore in conclusion f is convex for $x \in (-2, x_0) \cup (3, +\infty)$ and concave for $x \in (-\infty, -2) \cup (x_0, 3)$. Moreover, since $f''(x_0) = 0$ the function f exhibit an inflection point (with oblique tangent) at the point $(x_0, f(x_0)) \approx (0,69, 0,053)$.

8. **Graph.**
   Thanks to all the information we gained in points (1)-(7) we are now able to draw the graph of the function f.

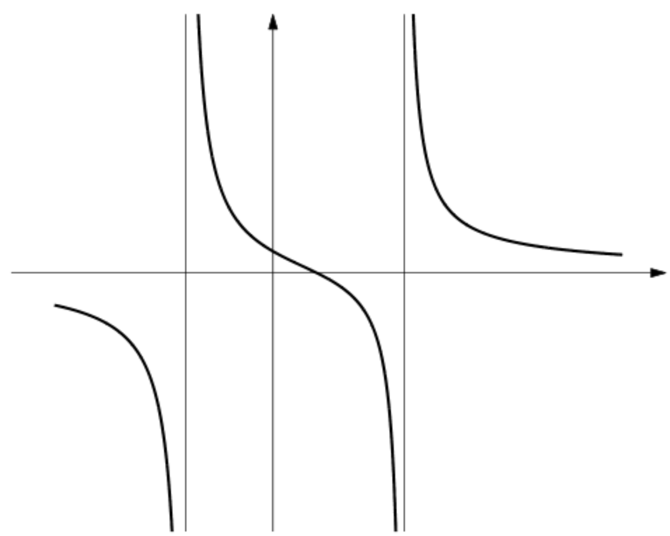

Figure 3:

2.2 Example 2

Consider the function

$$f(x) = e^{-x}(1 - e^{-2x})$$

1. **Domain.**
   The exponential function is always defined so there are no condition on x and the domain is the whole

real line: $D_f = \mathbb{R}$.

2. **Simmetries.**
   Calculating $f(-x)$ we have
$$f(-x) = e^{-x}(1 - e^{-2x})$$
   that is obviously different from $f(x)$ and $-f(x)$ too. So f is neither even nor odd and does not exhibit simmetries.

3. **Sign of the function.**
   Remember that an exponential is always positive in its domain, thus $e^{-x} > 0$ for every $x \in \mathbb{R}$ and we are left with the study of the sign of $1 - e^{-2x}$. Solving
$$1 - e^{-2x} \geq 0 \quad \Rightarrow \quad e^{-2x} \leq 1 \quad \Rightarrow \quad -2x \leq 0 \quad \Rightarrow \quad x \geq 0$$
   we conclude that f is positive for $x > 0$, negative for $x < 0$ and that $f(0) = 0$.

4. **Intersection with the axis.**
   Thanks to the above calculation we know that the only intersection with the axis occour at the origin.

5. **Limits and asymptotes.**
   The only relevant limits are the ones at $x \to \pm\infty$.
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{-x}(1 - e^{-2x}) = [\infty \cdot (1 - \infty)] = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^{-x}(1 - e^{-2x}) = [0 \cdot (1 - 0)] = 0$$
   therfore f has a (right) horizontal asymptote as $x \to +\infty$. On the other hand, as $x \to -\infty$ there are no horizontal asymptotes but there may be an oblique one. Trying to calculate the angular coefficient of this asymptote
$$m = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{e^{-x}(1 - e^{-2x})}{x} = \lim_{x\to-\infty} \frac{e^{-3x}}{x} = -\infty$$
   we conclude that does not exist such a straight line. Note that the above limit has been calculated thanks to hierarchy of the infinite (because an exponential always goes to infinity faster than a polynominal, so it is as if there was a bigger infiniti at the numerator.

6. **Monotonicity (first derivative).**
   We have
$$f'(x) = -e^{-x}(1 - e^{-2x}) + 2e^{-2x}e^{-x} = -e^{-x} + e^{-3x} + 2e^{-3x} = -e^{-x}(1 - 3e^{-2x})$$
   and since $-e^{-x}$ is always negative, we conclude that $f'(x) \geq 0$ if $1 - 3e^{-2x} \leq 0$ that is
$$3e^{-2x} \geq 1 \quad \Rightarrow \quad -2x \geq \ln\frac{1}{3} \quad \Rightarrow \quad -2x \geq -\ln 3 \quad \Rightarrow \quad x \leq \frac{\ln 3}{2}$$
   The table is as follows: and we infer that $x_0 = \frac{\ln 3}{2} \approx 0,55$ is a maximum point. The actual (global)

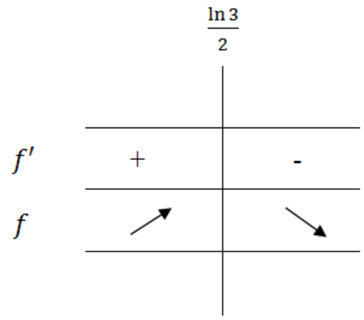

Figure 4:

maximum is achieved at x_0 and its value is

$$f\left(\frac{\ln 3}{2}\right) = e^{-\frac{\ln 3}{2}}(1 - e^{-\ln 3}) = (e^{\ln 3})^{-\frac{1}{2}}\left(1 - \frac{1}{3}\right) = \frac{2}{3} \cdot 3^{-\frac{1}{2}} = \frac{2}{3\sqrt{3}} \approx 0,38$$

7. **Convexity (second derivative).**
We have

$$f''(x) = e^{-x}(1 - 3e^{-2x}) - 6e^{-2x}e^{-x} = e^{-x} + -3e^{-3x} - 6e^{-3x} = e^{-x}(1 - 9e^{-2x})$$

As before, since $e^{-x} > 0$ for every $x \in \mathbb{R}$, the sign of f'' is determined by the sign of $1 - 9e^{-2x}$. In particular, $f'' \geq 0$ if

$$1 - 9e^{-2x} \geq 0 \quad \Rightarrow \quad e^{-2x} \leq \frac{1}{9} \quad \Rightarrow \quad -2x \leq -\ln 9 \quad \Rightarrow \quad x \geq \frac{\ln 9}{2} = \ln 3$$

The table is as follows:

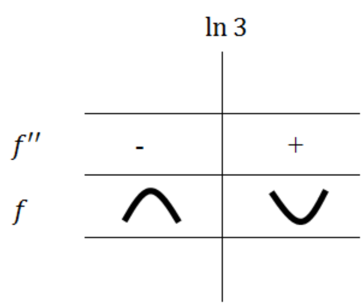

Figure 5:

Therefore in conclusion f is convex for $x \in (\ln 3, +\infty)$ and concave for $x \in (-\infty, \ln 3)$. Moreover, since $f''(\ln 3) = 0$ the function f exhibit an inflection point (with oblique tangent) at the point

$$\left(\ln 3, \frac{8}{27}\right) \approx (1.1, 0.3)$$

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8. **Graph.**

Thanks to all the information we gained in points (1)-(7) we are now able to draw the graph of the function f.

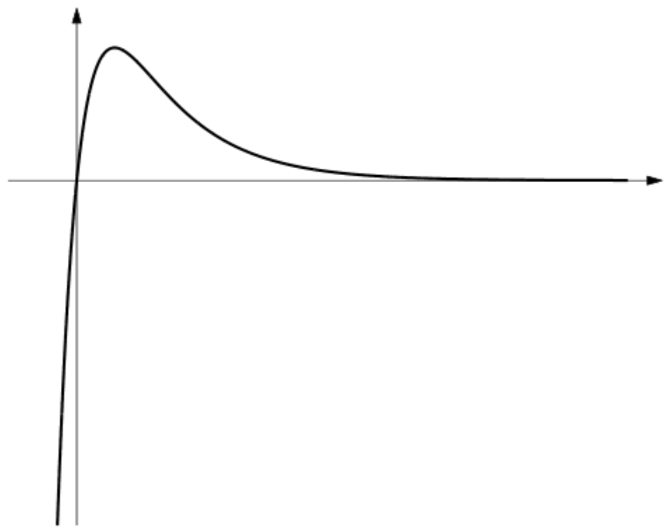

Figure 6: